

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1290

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Translation

"Struttura della funzione di Airy nei sistemi molteplicemente
connessi." *Giornale di Matematiche*, Vol. 77, 1947.



Washington

July 1951



NACA TM 1290

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THE STRUCTURE OF AIRY'S STRESS FUNCTION IN MULTIPLY
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The determination of the stress state in a plane, homogeneous, and isotropic elastic system with assigned forces on the boundary and in the absence of body forces¹ is known to lead, for the cases of both plane forces and strains, to the investigation of a biharmonic function, namely, the Airy function.

With the aid of this function, various investigations have been conducted, particularly on elasticity problems of plane simply or doubly connected stress systems.

In attempting to establish a systematic treatment of the problem of elastic equilibrium of a plane homogeneous and isotropic system of any shape and possessing any order of connectivity, the first step is to determine the singular terms (absent in a simply connected system) of Airy function in the most general case of stress and strain; on the basis of their significance, it may be decided which of these terms remain in various applied cases.

It is therefore proposed to isolate the singularities of the Airy function for a general plane system and to show how these functions are connected by their mechanical properties with the state of strain and stress of the elastic system.

*"Struttura della funzione di Airy nei sistemi molteplicemente connessi." *Giornale di Matematiche*, Vol. 77, 1947, pp. 119-144.

¹In cases of interest in which body forces are present, as, for example, for constant body forces (gravity forces), the problem may be reduced to one without these forces by changing the unknown function.

A special decomposition of the Airy function is therefore given; for simplicity of presentation, the following two different cases will be distinguished:

(a) systems subjected to a plane dislocation and therefore to a state of stress with zero forces on the boundary

(b) systems subjected to an external stress of a general type, but in the absence of dislocation (one-valued displacements)

It is evident that from the two compositions corresponding to cases (a) and (b) the one is immediately obtained that corresponds to the general case of a plane system in which the state of deformation is the resultant of those deformations due to a general external stress and a plane dislocation.

For a plane dislocation, only the components of the displacement parallel to the plane of the system are considered herein. Such a displacement results when, after the cuts that decrease the order of connectivity of the system are made, one of the faces of each single cut is given a displacement relative to the other faces. The displacement is composed of a translation parallel to the plane of the system and a rotation about an axis perpendicular to this plane; a certain quantity of material is then added or subtracted, as the case may be².

Treatment of the problems of plane dislocations with the aid of the Airy function seems natural, because the determination of the states of strain and of stress of a system subjected only to dislocation leads essentially to the problem of the integration of the equations of elastic equilibrium with assigned forces at the boundary.

In the two decompositions presented, some of the singular parts figure directly as known elements in that their coefficients may be expressed in terms of assigned elements characteristic of the state of stress and strain to which the system is subjected. That is, among the coefficients of the singular terms, the characteristics of the dislocation figure in case (a); whereas in case (b), the components of the resultant forces and the resultant moments of the external stresses acting on the boundaries of the individual holes are involved.

²Results on the plane dislocations of a circular ring will be found in references 1 and 2.

For the first case, the expression given by the Airy function satisfies the condition of many valuedness of the required displacements from the dislocation considered; the condition of single-valuedness of the displacements is satisfied in the second case. The analytical problem that remains to be solved therefore reduces in each case to a boundary problem similar (but not identical) to that to which simply connected systems are reduced.

It was also desired to show that even those coefficients of the singular terms that are not directly known are expressible in terms of characteristic elements of the deformation and that their invariance with respect to certain curves, with reference to which they are defined, corresponds to mechanical properties common to all continuous systems at rest, the known properties of the mean stress.

In order to determine the character of the single-valued part of the Airy function in the two cases considered, a special decomposition of the biharmonic functions is employed in the manner of Poincaré. This decomposition was established several years ago by G. Fichera (reference 3) and was found to be very useful for the purpose.

The many-valued part was determined by direct considerations, but it is necessarily present if, and only if, the acting forces on each singular boundary do not constitute a system in equilibrium.

From the given decomposition for the Airy function, those components of the displacement and of the rotation from which the singularities appear are derived without difficulty. It is thus possible to treat systematically also problems concerning systems in which the displacements as well as the forces are assigned on the boundary or partly the displacements and partly the forces (mixed problems). Such decomposition formulas are given subsequently.

For simplicity, a doubly connected system will first be presented; the results will then be extended to a system of any order of connectivity.

The essential concepts of the theory of elastic dislocations that is used herein are directly derived from the fundamental investigations of Vito Volterra³.

³Published in various notes in the reports of the R. Accademia dei Lincei in the years 1905-1906 and collected in one memoir in reference 4.

The author wishes to express his sincere thanks to Signora Virginia Volterra, widow of Vito Volterra and to his son Prof. Enrico Volterra who kindly furnished him with the Italian and French editions of the papers.

1. Many-Valuedness of Airy Function

Assume the plane system defined in a region T bounded externally by the curve C_0 , internally by the curve C_1 and referred to a pair of orthogonal axes Ox, Oy with origin inside the area enclosed by C_1 .

For the Airy function $F(x,y)$ the following formulas are well known:

$$X_x = \frac{\partial^2 F}{\partial y^2} \quad Y_y = \frac{\partial^2 F}{\partial x^2} \quad X_y = Y_x = - \frac{\partial^2 F}{\partial x \partial y} \quad (1)$$

with the usual significance of the symbols.

For any curve l the tangent and normal vectors denoted by t and n , respectively, are oriented so that the pair t, n is superposable in a rigid displacement of the plane on the oriented pair, $-x, y$.

There is then obtained

$$n_x = \frac{dy}{ds} \quad n_y = - \frac{dx}{ds} \quad (2)$$

If X_n, Y_n denote the components of the specific force corresponding to any point of l and in the oriented direction of n ,⁴ from the formulas

$$\left. \begin{aligned} X_n &= X_x n_x + X_y n_y \\ Y_n &= X_y n_x + Y_y n_y \end{aligned} \right\} \quad (3)$$

⁴Specifically, X_n, Y_n refer to the actions that the molecules situated on the issuing side of the boundary exert upon the molecules on the other side.

and from equations (1) and (2) there is immediately obtained

$$\left. \begin{aligned} X_n &= \frac{d}{ds} \left(\frac{\partial F}{\partial y} \right) \\ Y_n &= - \frac{d}{ds} \left(\frac{\partial F}{\partial x} \right) \end{aligned} \right\} \quad (4)$$

If M_n denotes the moment with respect to the third axis of coordinates z , of the vector of components X_n, Y_n , there follows from equations (1) to (3)

$$M_n = x Y_n - y X_n = \frac{d}{ds} \left[F - x \frac{\partial F}{\partial x} - y \frac{\partial F}{\partial y} \right] \quad (5)$$

In the following discussion, c will denote any closed curve contained in T and inclosing C_1 . When n is identified with the outer normal, the positive direction of transversing the curve, that is, the direction of t , is then such as to leave the area to the left.

By making l coincide with c there is obtained from equations (4) and (5)⁵

$$\left. \begin{aligned} \int_c d \left(\frac{\partial F}{\partial y} \right) &= - X \\ \int_c d \left(\frac{\partial F}{\partial x} \right) &= Y \end{aligned} \right\} \quad (6)$$

$$\int_c d \left[F - x \frac{\partial F}{\partial x} - y \frac{\partial F}{\partial y} \right] = - M \quad (7)$$

⁵Formulas (6) and (7) are in accordance with a property of the Airy function already brought to light. (See reference 5.)

where X and Y denote the components of the resultant of the forces transmitted across the curve c from the interior toward the exterior and M is their resultant moment with respect to z .

In the absence of body forces, however, X, Y, M are the corresponding components of the resultant force and moment of the external stress acting on the boundary C_1 of the plane system. It is clearly shown in equations (6) and (7) that, in the presence of external stress of a quite general type, the expressions contained under the differential sign are necessarily many-valued. It follows that the function $F(x, y)$ and the first derivatives will in general be many-valued, but second derivatives will be single-valued (on account of their significance as stresses).

If the two functions satisfying equations (6) and (7) are denoted by ψ_1, ψ_2 , the function $\psi_1 - \psi_2$, together with its derivatives, is necessarily single-valued, which is sufficient to assure that F is of the type

$$F(x, y) = \varphi(x, y) + \psi'(x, y) \quad (8)$$

with $\varphi(x, y)$ single-valued and $\psi'(x, y)$ uniquely determined from equations (6) and (7).

When $\theta = \arctan y/x$, the many-valued function $-M\theta/2\pi$ (harmonic and with single-valued derivatives) satisfies equation (7). It is then found immediately that $\psi'(x, y)$ corresponds to the product of θ by a linear function of x, y , the coefficients of which are determined by taking into account equations (6) and (7).

There is thus found

$$\psi'(x, y) = -\frac{M_p}{2\pi} \theta \quad (9)$$

where

$$M_p = M + yX - xY \quad (10)$$

is the resultant moment with respect to the pole $P(x, y)$ of the external forces applied to C_1 .

On the basis of equations (8) and (9), the Airy function can now be given the form

$$F(x,y) = \varphi(x,y) - \frac{M_p}{2\pi} \theta \quad (11)$$

2. Decomposition of the Single-Valued Part of the Airy Function

It will be very useful in the following discussion to adopt for the function $\varphi(x,y)$ the decomposition formula of Poincaré (see paper by Fichera already referred to) and to write

$$\begin{aligned} \varphi(x,y) = \varphi_0(x,y) + \varphi_1(x,y) + \left[\alpha \rho^2 + \beta x + \gamma y + \delta \right] \log \rho + \\ a \cos 2\theta + b \sin 2\theta \end{aligned} \quad (12)$$

The functions $\varphi_0(x,y)$, $\varphi_1(x,y)$ are biharmonic, the first regular at all points not external to the region bounded by C_0 , and the second regular at all points not internal to the region bounded by C_1 and converging at infinity.

In agreement with the notation adopted in reference 3 referred to,

$$\Phi[u,v;c] = \frac{1}{8\pi} \left\{ \int_c \left[v \frac{d\Delta u}{dn} - u \frac{d\Delta v}{dn} \right] ds + \int_c \left[\Delta v \frac{du}{dn} - \Delta u \frac{dv}{dn} \right] ds \right\} \quad (13)$$

where, if u,v are biharmonic in T , the value of $\Phi[u,v;c]$ does not depend on the curve c , and the constants α , β , γ , δ , a , and b are expressed by the formulas

$$\left. \begin{aligned} \alpha &= \Phi[\varphi, 1; c] \\ \beta &= -2\Phi[\varphi, x; c] \\ \gamma &= -2\Phi[\varphi, y; c] \end{aligned} \right\} \quad \left. \begin{aligned} \delta &= \Phi[\varphi, \rho^2; c] \\ a &= \frac{1}{2}\Phi[\varphi, \rho^2 \cos 2\theta; c] \\ b &= \frac{1}{2}\Phi[\varphi, \rho^2 \sin 2\theta; c] \end{aligned} \right\} \quad (14)$$

3. Expressions of Components of Displacement

If e_x, e_y, e_{xy} are the strains, ω the rotation, u, v the components of the displacement, following well-known formulas⁶ may be written:

$$\left. \begin{aligned} \frac{\partial u}{\partial x} &= e_x & \frac{\partial u}{\partial y} &= \frac{1}{2} e_{xy} - \omega & \frac{\partial \omega}{\partial x} &= \frac{1}{2} \frac{\partial e_{xy}}{\partial x} - \frac{\partial e_x}{\partial y} \\ \frac{\partial v}{\partial x} &= \frac{1}{2} e_{xy} + \omega & \frac{\partial v}{\partial y} &= e_y & \frac{\partial \omega}{\partial y} &= \frac{\partial e_y}{\partial x} - \frac{1}{2} \frac{\partial e_{xy}}{\partial y} \end{aligned} \right\} \quad (15)$$

where e_x, e_y, e_{xy} are connected with X_x, Y_y, X_y by the relations⁷

$$\left. \begin{aligned} e_x &= \frac{k}{E} (X_x - \tau Y_y) \\ e_y &= \frac{k}{E} (Y_y - \tau X_x) \\ e_{xy} &= 2 \frac{1+\nu}{E} X_y \end{aligned} \right\} \quad (16)$$

where

$$k \begin{cases} = 1 & \text{in the case of plane stress} \\ = 1-\nu^2 & \text{in the case of plane strain} \end{cases} \quad (17)$$

$$\tau \begin{cases} = \nu & \text{in the case of plane stress} \\ = \nu/(1-\nu^2) & \text{in the case of plane strain} \end{cases} \quad (18)$$

⁶(See, for example, reference 6.)

⁷(See, for example, reference 7.)

In equations (16) to (18), ν denotes Poisson's ratio and E , Young's modulus.

From equations (1), (15), and (16) there follows immediately

$$\left. \begin{aligned} \frac{\partial u}{\partial x} &= \frac{k}{E} \left(\frac{\partial^2 F}{\partial y^2} - \tau \frac{\partial^2 F}{\partial x^2} \right) & \frac{\partial u}{\partial y} &= - \frac{1+\nu}{E} \frac{\partial^2 F}{\partial x \partial y} - \omega \\ \frac{\partial v}{\partial x} &= - \frac{1+\nu}{E} \frac{\partial^2 F}{\partial x \partial y} + \omega & \frac{\partial v}{\partial y} &= \frac{k}{E} \left(\frac{\partial^2 F}{\partial x^2} - \tau \frac{\partial^2 F}{\partial y^2} \right) \end{aligned} \right\} \quad (19)$$

and

$$\begin{aligned} \frac{\partial \omega}{\partial x} &= - \frac{k}{E} \frac{\partial \Delta F}{\partial y} \\ \frac{\partial \omega}{\partial y} &= \frac{k}{E} \frac{\partial \Delta F}{\partial x} \end{aligned} \quad (20)$$

By denoting now as l any line internal to T that unites the points $P_0 \equiv (x_0, y_0)$ and $P \equiv (x, y)$, the following expression may be written:

$$\int_l \omega d\eta = \eta \omega - \eta_0 \omega_0 - \int_l \eta d\omega \quad (21)$$

where η denotes either of the variables x, y .

Bearing in mind by means of equations (19) to (21), there is obtained with some transformations

$$\left. \begin{aligned}
 u(x,y) &= u(x_0,y_0) - [y \omega(x,y) - y_0 \omega(x_0,y_0)] + \\
 &\frac{k}{E} \left\{ \int_{\gamma} \left[y \frac{d\Delta F}{dn} - \Delta F \frac{dy}{dn} \right] ds - (\tau+1) \int_{\gamma} d \left(\frac{\partial F}{\partial x} \right) \right\} \\
 v(x,y) &= v(x_0,y_0) + [x \omega(x,y) - x_0 \omega(x_0,y_0)] + \\
 &+ \frac{k}{E} \left\{ - \int_{\gamma} \left[x \frac{d\Delta F}{dn} - \Delta F \frac{dx}{dn} \right] ds - (\tau+1) \int_{\gamma} d \left(\frac{\partial F}{\partial y} \right) \right\} \\
 \omega(x,y) &= \omega(x_0,y_0) + \frac{k}{E} \int_{\gamma} \frac{d\Delta F}{dn} ds
 \end{aligned} \right\} \quad (22)$$

4. Airy Function Corresponding to Plane Dislocation in Doubly Connected Field

With reference to case (a) of the introduction, the system is now assumed subjected to a state of strain due to a plane dislocation. In the decomposition of the Airy function defined by equations (10) to (12), the constants α , β , and γ will be shown to represent the three characteristics of the plane dislocation as determined by the latter equation.

In order to show this relation, it is first observed that from equation (22) the necessary and sufficient conditions for the single valuedness of the components of the displacement and of the rotation are clearly expressed by

$$\left. \begin{aligned} \int_C \frac{d\Delta F}{dn} ds &= 0 \\ \int_C \left[y \frac{d\Delta F}{dn} - \Delta F \frac{dy}{dn} \right] ds - (\tau+1) \int_C d \left(\frac{\partial F}{\partial x} \right) &= 0 \\ \int_C \left[x \frac{d\Delta F}{dn} - \Delta F \frac{dx}{dn} \right] ds + (\tau+1) \int_C d \left(\frac{\partial F}{\partial y} \right) &= 0 \end{aligned} \right\} \quad (23)$$

The functions $\varphi_0(x,y)$ and $\varphi_1(x,y)$ from their nature⁸ satisfy equation (23).

Because the functions of $\log \rho$ and θ are also harmonic and moreover because $\log \rho$ is single valued with its derivatives and θ is many valued, but with first derivatives single-valued, the two terms behave similarly.

It is not difficult to show, by substituting, that also the functions $\cos 2\theta$ and $\sin 2\theta$ satisfy equations (23).

⁸This is clear for equation (23.1) on the basis of a known theorem, if it is assumed that the functions $\Delta\varphi_0$ and $\Delta\varphi_1$ are harmonic. As regards equation (23.2), it is noted that on the basis of equations (2),

$$\left(y \frac{d\Delta F}{dn} - \Delta F \frac{dy}{dn} \right) ds = y \frac{\partial \Delta F}{\partial x} dy - \left(y \frac{\partial \Delta F}{\partial y} - \Delta F \right) dx$$

from which, by taking into account the biharmonic character of F the expression on the left of this equation is proved an exact differential. The left member of equation (23.2) is of the form of integrals of exact differentials. From the fact that φ_0 has for its field of regularity the region enclosed by C_0 , it is then concluded that the left side of equation (23.2) vanishes for $F \equiv \varphi_0$. The same result is obtained for $F \equiv \varphi_1$ if it is considered that its field of regularity is composed of all the points of the plane not internal to the region enclosed by C_1 . Similar considerations apply to equation (23.3).

Because of the properties of the terms that enter in the expression of the Airy function (see equations (10) to (12)), only those contained in the expression

$$F^*(x,y) = (\alpha \rho^2 + \beta x + \gamma y) \log \rho + \frac{1}{2\pi} [x Y - y X] \theta \quad (24)$$

give rise to many-valued displacements.

The increments received by $u(x,y)$, $v(x,y)$, and $\omega(x,y)$ are denoted by u^* , v^* , and ω^* , respectively. When a complete circuit (started from the point of coordinates x,y) is effected about the hole, equations (22) that $E\omega^*/k$, $E(u^* + y\omega^*)/k$, $-E(v^* - x\omega^*)/k$ agree with the values assumed by the first members of equation (23) for $F(x,y) \equiv F^*(x,y)$.

If this result is taken into account, it is found from equations (23) and (24) that

$$\left. \begin{aligned} u^* &= -2\pi \frac{k}{E} \left[4\alpha y + 2\gamma - \frac{1-\tau}{2\pi} Y \right] \\ v^* &= 2\pi \frac{k}{E} \left[4\alpha x + 2\beta - \frac{1-\tau}{2\pi} X \right] \\ \omega^* &= 8\pi \frac{k}{E} \alpha \end{aligned} \right\} \quad (25)$$

Because a state of strain with surface forces (except for body forces) at the boundaries corresponds to each dislocation, there must be set $X = Y = 0$. From equations (25) the result is then obtained: The three characteristics of the plane dislocation (constants of the cut) are given by

$$\begin{aligned} &-4\pi \frac{k}{E} \gamma \\ &4\pi \frac{k}{E} \beta \\ &-8\pi \frac{k}{E} \alpha \end{aligned}$$

The first two characteristics correspond to parallel fissures and the third to a radial fissure⁹.

These fissures are assumed to be assigned on the basis of the particular dislocation considered so that, when they are denoted by l , m , and r , the following expression is obtained: Corresponding to a general plane dislocation of characteristics l , m , and r the Airy function in a doubly connected field is capable of the decomposition

$$F = \varphi_0 + \varphi_1 + \frac{E}{4\pi k} \left[-\frac{r}{2} \rho^2 + mx - ly \right] \log \rho + \delta \log \rho + a \cos 2\theta + b \sin 2\theta \quad (26)$$

The unknown part $\bar{F} = F - F^*$ of the right-hand side of equation (26) is uniquely determined by the condition that the forces on the complete boundary of the plane system are zero.

Explicitly on the basis of equation (4) this condition is given by the equations

$$\left. \begin{aligned} \frac{\partial \bar{F}}{\partial x} &= -\frac{\partial F^*}{\partial x} + \text{const.} \\ \frac{\partial \bar{F}}{\partial y} &= -\frac{\partial F^*}{\partial y} + \text{const.} \end{aligned} \right\} \text{ on } C_0 \text{ and } C_1$$

⁹It is evident that these definitions give an immediate generalization of those used in the work of V. Volterra in the case of a field having the form of a circular ring. That is, the dislocation will be said to correspond to a uniform fissure when the displacement that generates it is a translation and to a radial fissure when the displacement is a rotation.

5. Airy Function in the Case of Single-Valued Displacements in Doubly Connected Field

All the elements are now at hand for determining the structure of the Airy function in the absence of dislocations corresponding to any external stress. The coefficients α , β , and γ will now also be determined. In fact, the conditions of single-valuedness in equation (23), which are more synthetically expressed by

$$u^* = 0, \quad v^* = 0, \quad \omega^* = 0$$

on the basis of equation (25) lead to

$$\alpha = 0, \quad \gamma = (1-\tau)Y/4\pi, \quad \beta = (1-\tau)X/4\pi$$

From these relations and from equations (10) to (12), the following result is obtained: The Airy function in the case of single-valued displacements in a doubly connected field is capable of the decomposition

$$F = \varphi_0 + \varphi_1 + \delta \log \rho + a \cos 2\theta + b \sin 2\theta + \frac{1}{2\pi} \left\{ (X x + Y y) \frac{1-\tau}{2} \log \rho - M_p \theta \right\} \quad (27)$$

Remark. It is sufficient to glance at equation (27) in order to derive the necessary and sufficient condition that the Airy function of a doubly connected system free from restraints does not depend, in the absence of dislocations, on the elastic constants of the system and that the external forces applied at the boundary of the hole have a zero resultant. The same is naturally true for the stresses.

6. Structure of the Airy Function in Multiply Connected Systems

The preceding considerations will now be extended to the case of a plane system of any number of connections. The doubly connected system already considered will be referred to a pair of parallel axes coinciding with those already adopted but having their origin at a general point of coordinates $-\xi, -\eta$, which may be external to the whole. It is evident that the previously derived formulas, if referred to moving axes (denoted by x, y), are written by putting in place of x, y , respectively, $x-\xi, y-\eta$.

In particular the Airy function will have the expression (see equations (10) to (12))

$$F = \varphi_0 + \varphi_1 + \left[\alpha \rho^{*2} + \beta^* (x-\xi) + \gamma^* (y-\eta) + \delta^* \right] \log \rho^* + a^* \cos 2\theta + b^* \sin 2\theta - \frac{1}{2\pi} M_p \theta^* \quad (28)$$

where ρ^* is the distance between the points of coordinate x, y and ξ, η and θ is $\arctan \frac{y-\eta}{x-\xi}$. It should be remarked that the point of coordinates ξ, η is interior to the hole. The constants $\alpha^*, \beta^*, \gamma^*, \delta^*, a^*$, and b^* are defined by the same equations (14) except that $x-\xi, y-\eta$ are substituted for x and y and ρ^* for ρ .

Equation (28) gives a decomposition of the Airy function referred to a pair of axes x, y completely arbitrary but with ξ, η having the significance of coordinates of a point Q within the hole; this equation is generalized to the case of plane systems with any order of connectivity.

The region of definition of the Airy function will now be considered as a region T bounded by the curve C_0 and having n holes bounded by the curves C_1, C_2, \dots, C_n ; ξ_i, η_i ($i = 1, 2, \dots, n$) will be taken as the coordinates of a point Q_i within the hole bounded by C_i ($i = 1, 2, \dots, n$).

Setting

$$\left. \begin{aligned} \rho_i^2 &= (x - \xi_i)^2 + (y - \eta_i)^2 \\ \theta_i &= \arctan \frac{y - \eta_i}{x - \xi_i} \end{aligned} \right\}$$

the Airy function in this region, by evident generalization of equation (28), has the following form

$$F = \varphi_0 + \sum_{i=1}^n \varphi_i + \sum_{i=1}^n \left\{ \left[\alpha_i \rho_i^2 + \beta_i (x - \xi_i) + \gamma_i (y - \eta_i) + \delta_i \right] \log \rho_i + a_i \cos 2\theta_i + b_i \sin 2\theta_i \right\} - \frac{1}{2\pi} \sum_{i=1}^n M_p^{(i)} \theta_i \quad (29)$$

In this equation, φ_0 is biharmonic and is regular in the manifold bounded by C_0 ; φ_i ($i = 1, 2, \dots, n$) is biharmonic outside the hole bounded by C_i ($i = 1, 2, \dots, n$) and converges at infinity.

If X_i , Y_i ($i = 1, 2, \dots, n$) are the components of the resultant of the external forces applied at the boundary of the hole bounded by C_i and M_i is their resultant moment with respect to the axis oriented as z through Q_i , $M_p^{(i)} = (y - \eta_i)X_i - (x - \xi_i)Y_i + M_i$ denotes the resultant moment applied at C_i with respect to the axis oriented as z through the point $P \equiv (x, y)$. The constants α_i , β_i , γ_i , δ_i , a_i , and b_i are determined by the formulas

$$\left. \begin{aligned} \alpha_i &= \Phi [\varphi, 1; c_i] \\ \beta_i &= -2\Phi [\varphi, x - \xi_i, c_i] \\ \gamma_i &= -2\Phi [\varphi, y - \eta_i, c_i] \\ \delta_i &= \Phi [\varphi, \rho_i^2; c_i] \\ a_i &= \frac{1}{2}\Phi [\varphi, \rho_i^2 \cos 2\theta_i; c_i] \\ b_i &= \frac{1}{2}\Phi [\varphi, \rho_i^2 \sin 2\theta_i; c_i] \end{aligned} \right\} \quad (30)$$

where Φ denotes the single-valued part of the Airy function.

It will now be shown that in equation (29) the coefficient¹⁰ of $\log \rho_i$ does not depend on $\xi_i, \eta_i, (i = 1, 2, \dots, n)$.

Because of the linearity of the operator Φ , the following expression may be written, for any value of i

$$\beta_i = -2 \Phi [\varphi, x; c_i] + 2 \xi_i \Phi [\varphi, 1; c_i] = \bar{\beta}_i + 2 \alpha_i \xi_i \quad (31)$$

where

$$\bar{\beta}_i = -2 \Phi [\varphi, x; c_i] \quad (32)$$

Similarly

$$\gamma_i = \bar{\gamma}_i + 2 \alpha_i \eta_i \quad (33)$$

with

$$\bar{\gamma}_i = -2 \Phi [\varphi, y; c_i] \quad (34)$$

Moreover

$$\begin{aligned} \delta_i = & \Phi [\varphi, \bar{\rho}^2; c_i] + (\xi_i^2 + \eta_i^2) \Phi [\varphi, 1; c_i] - \\ & 2 \xi_i \Phi [\varphi, x; c_i] - 2 \eta_i \Phi [\varphi, y; c_i] \end{aligned} \quad (35)$$

where $\bar{\rho}$ denotes the distance from the origin of any point $P \equiv (x, y)$.

From equations (30.1), (32), (34), and (35), the following expression is immediately obtained

$$\delta_i = \bar{\delta}_i + \alpha_i (\xi_i^2 + \eta_i^2) + \bar{\beta}_i \xi_i + \bar{\gamma}_i \eta_i \quad (36)$$

¹⁰A similar observation can evidently be made with regard to the coefficient of $\log \rho^*$ in equation (28).

with

$$\bar{\delta} = \Phi \left[\varphi, \bar{\rho}^2; c_i \right] \quad (37)$$

On the basis of equations (31), (33), and (36), the following equation can be written:

$$\alpha_i \rho_i^2 + \beta_i (x - \xi_i) + \gamma_i (y - \eta_i) + \delta_i = \alpha_i \bar{\rho}^2 + \bar{\beta}_i x + \bar{\gamma}_i y + \bar{\delta}_i \quad (38)$$

Equation (29) can therefore be written as

$$F = \varphi_0 + \sum_{i=1}^n \varphi_i + \sum_{i=1}^n \left\{ \left[\alpha_i \bar{\rho}^2 + \bar{\beta}_i x + \bar{\gamma}_i y + \bar{\delta}_i \right] \log \rho_i + a_i \cos 2\theta_i + b_i \sin 2\theta_i - \frac{1}{2\pi} M_p^{(i)} \theta_i \right\} \quad (39)$$

The meaning of $\alpha_i, \bar{\beta}_i, \bar{\gamma}_i$ remains to be determined. For this purpose, u_s^*, v_s^*, ω_s^* ($s = 1, 2, \dots, n$) denote the increments received by u, v, ω when a complete turn is made about the hole bounded by C_s ($s = 1, 2, \dots, n$) starting from the point $P \equiv (x, y)$. It is evident that, of the terms of the expression equation (29) of the Airy function, only those of index s contained in the various sums can make a nonzero contribution to the formation u_s^*, v_s^*, ω_s^* ($s = 1, 2, \dots, n$), because the others are regular with all the derivatives in the region bounded by C_s ($s = 1, 2, \dots, n$).

Substantially, the same relation holds true as in a double connected system and the expressions of u_s^*, v_s^*, ω_s^* are obtained from those of u^*, v^*, ω^* (equation (25)) by simply substituting $\alpha_s, \beta_s, \gamma_s, X_s, Y_s$ for $\alpha, \beta, \gamma, X, Y$, respectively, and $x - \xi_s, y - \eta_s$ for x, y .

$$\left. \begin{aligned} u_s^* &= -2\pi \frac{k}{E} \left[4 \alpha_s (y - \eta_s) + 2 \gamma_s - \frac{1 - \tau}{2\pi} Y_s \right] \\ v_s^* &= 2\pi \frac{k}{E} \left[4 \alpha_s (x - \xi_s) + 2 \beta_s - \frac{1 - \tau}{2\pi} X_s \right], \quad (s = 1, 2, \dots, n) \\ \omega_s^* &= 8\pi \frac{k}{E} \alpha_s \end{aligned} \right\} \quad (40)$$

It is then found that u_s^* , v_s^* , ω_s^* do not depend on the points (ξ_s, η_s) .

In fact, on the basis of equations (31) and (33), equations (40) may be written

$$\left. \begin{aligned} u_s^* &= -2\pi \frac{k}{E} \left[4 \alpha_s y + 2 \bar{\gamma}_s - \frac{1 - \tau}{2\pi} Y_s \right], \\ v_s^* &= 2\pi \frac{k}{E} \left[4 \alpha_s x + 2 \bar{\beta}_s - \frac{1 - \tau}{2\pi} X_s \right], \quad (s = 1, 2, \dots, n) \\ \omega_s^* &= 8\pi \frac{k}{E} \alpha_s \end{aligned} \right\} \quad (41)$$

from which immediately follows the result: If l_i , m_i , and r_i are the characteristics of a plane dislocation corresponding to a cut that goes from C_0 to C_i ($i = 1, 2, \dots, n$),

$$\left. \begin{aligned} \alpha_i &= -\frac{E}{8\pi k} r_i \\ \bar{\beta}_i &= \frac{E}{4\pi k} m_i \\ \bar{\gamma}_i &= -\frac{E}{4\pi k} l_i \end{aligned} \right\} \quad (42)$$

It is also clear that the $\bar{\beta}_i$, $\bar{\gamma}_i$ correspond to parallel fissures and the α_i to radial fissures.

From equations (38) and (42) the final result is thus obtained: The structure of the Airy function corresponding to a general plane dislocation in an $(n+1)$ -fold connected field is defined by the formula:

$$F = \varphi_0 + \sum_{i=1}^n \varphi_i + \sum_{i=1}^n \left\{ \frac{E}{4\pi k} \left[-\frac{r_i}{2} \rho^{-2} + m_i x - l_i y \right] + \bar{\delta}_i \right\} \log \rho_i + \sum_{i=1}^n [a_i \cos 2\theta_i + b_i \sin 2\theta_i]$$

In order that the displacements be single-valued, it is evidently necessary and sufficient that

$$\left. \begin{aligned} u_i^* &= 0 \\ v_i^* &= 0 \\ \omega_i^* &= 0 \end{aligned} \right\} \quad (i = 1, 2, \dots, n)$$

From equation (41): The necessary and sufficient condition for the single-valuedness of the displacement in an $(n+1)$ -fold connected plane system is that the following equalities be satisfied:

$$\left. \begin{aligned} \alpha_i &= 0 \\ \bar{\beta} &= \frac{1-\tau}{4\pi} X_i \\ \bar{\gamma} &= \frac{1-\tau}{4\pi} Y_i \end{aligned} \right\} \quad (i = 1, 2, \dots, n)$$

and therefore the Airy function in the absence of dislocations in an $(n+1)$ -fold connected field is capable of the decomposition¹¹

¹¹From (43) there clearly results the condition (analogous to that expressed in the remark of section 5) of the independence of F on the elastic constants of the system.

$$F = \varphi_0 + \sum_{i=1}^n \varphi_i + \sum_{i=1}^n \left[\bar{\delta}_i \log \rho_i + a_i \cos 2\theta_i + b_i \sin 2\theta_i \right] +$$

$$\frac{1}{2\pi} \sum_{i=1}^n \left\{ (X_i x + Y_i y) \frac{1-\tau}{2} \log \rho_i - M_p^{(i)} \theta_i \right\} \quad (43)$$

Remark. Equation (42) evidently also gives the expression of the Airy function in the presence of forces or moments concentrated at interior points of the plane system. If, for example, at the point $Q \equiv (\xi^*, \eta^*)$ there acts a concentrated stress (concentrated force or moment or both systems), one of the terms of the sum in equations (42) corresponds to it. If this term of the sum is, for example, of index 1, φ_1 is biharmonic at the exterior of each circle of center Q , and φ_1 converges at infinity, then $\bar{\delta}_1$, a_1 , b_1 are always defined by equations (37), (30.5), and (30.6), respectively. In these relations, c_1 is any circumference of center Q interior to C_0 and such as to leave the regions bounded by C_1, C_2, \dots on the outside; $X_1, Y_1, M_p^{(1)}$ denote the components of the resultant of the concentrated stress at Q and the resultant moment with respect to the straight line oriented as z through $P \equiv (x, y)$.

7. Mechanical Significance of Coefficients δ , a , b

As evident from the heading of the present section, the case of a doubly connected field is considered for simplicity, but the extension to the case of any number of connections will be evident.

The coefficients δ , a , b are not determined, as are α , β , and γ , by the condition of the problem; they nevertheless have a mechanical significance that is now given.

It will be shown that these coefficients are expressed in terms of characteristic elements of the deformation and of the stress such as the rotation, the coefficient of surface dilatation, and (only for the coefficient δ , however) the astatic coordinates of the external stress acting on one of the boundaries of the plane system.

It is first necessary to establish several preliminary formulas.
We set

$$F^{**} = \frac{1}{2\pi} [x Y - y X - M] \theta = - \frac{Mp}{2\pi} \theta \quad (44)$$

$$\left. \begin{aligned} \delta^{**} &= \Phi [F^{**}, \rho^2; c] \\ a^{**} &= \frac{1}{2} \Phi [F^{**}, \rho^2 \cos 2\theta; c] \\ b^{**} &= \frac{1}{2} \Phi [F^{**}, \rho^2 \sin 2\theta; c] \end{aligned} \right\} \quad (45)$$

From equations (44) and (45), if equation (13) is taken into account, there is obtained

$$\delta^{**} = \frac{1}{2\pi} [y_0 Y + x_0 X] \quad (46)$$

$$a^{**} = b^{**} = 0 \quad (47)$$

where x_0 , y_0 denote the coordinates of the point P_0 from which the path on c begins.

On the basis of equations (10), (11), (14), (44), and (45), and when the linearity of the operator Φ is taken into account, the following relation may be written:

$$\delta = \Phi [F, \rho^2; c] - \delta^{**} \quad (48)$$

where as when equation (47) is also accounted for,

$$\left. \begin{aligned} a &= \frac{1}{2} \Phi [F, \rho^2 \cos 2\theta; c] \\ b &= \frac{1}{2} \Phi [F, \rho^2 \sin 2\theta; c] \end{aligned} \right\} \quad (49)$$

Taking into account equations (2),

$$\int_c \frac{dF}{dn} ds = - \int_c \left(\frac{\partial F}{\partial y} dx - \frac{\partial F}{\partial x} dy \right) = - \int_c d \left(x \frac{\partial F}{\partial y} - y \frac{\partial F}{\partial x} \right) + \int_c \left(x d \frac{\partial F}{\partial y} - y d \frac{\partial F}{\partial x} \right) \quad (50)$$

From equation (50), on the basis of equations (4) and (6) there is immediately obtained

$$\int_c \frac{dF}{dn} ds = x_0 X + y_0 Y + \int_c (x X_n + y Y_n) ds \quad (51)$$

When two of the astatic coordinates (which two are required is evident) of the stress acting across c are denoted by a_{xx} and a_{yy} , equation (51) can be (see reference 8) written (see also equation (46))

$$\int_c \frac{dF}{dn} ds = 2\pi \delta^{**} + a_{xx} + a_{yy} \quad (52)$$

From equation (22.3) there is immediately derived

$$\frac{E}{k} d\omega = \frac{d\Delta F}{dn} ds \quad (53)$$

and also

$$\Delta F = \frac{E}{k(1-\tau)} D \quad (54)$$

where D denotes the divergence of the displacement u, v (coefficient of surface dilatation).

Mechanical significance of δ and of its invariance with respect to c . - When equation (48) is expressed explicitly, and on the basis of equation (13), the following expression may be written:

$$\delta = \frac{1}{8\pi} \left\{ \int_c \rho^2 \frac{d\Delta F}{dn} ds + 4 \int_c \frac{dF}{dn} ds - \int_c \Delta F \frac{d\rho^2}{dn} ds \right\} - \delta^{**}$$

Taking into account equations (52) to (54),

$$\delta = \frac{E}{8\pi k} \left\{ \int_c \rho^2 d\omega - \frac{1}{1-\tau} \int_c D \frac{d\rho^2}{dn} ds \right\} + \frac{1}{2\pi} (a_{xx} + a_{yy}) \quad (55)$$

or, taking into account equations (52) and (54)

$$\delta = \frac{E}{8\pi k (1-\tau)} \left\{ \int_c \rho^2 \frac{dD}{dn} ds - \int_c D \frac{d\rho^2}{dn} ds \right\} + \frac{1}{2\pi} (a_{xx} + a_{yy}) \quad (56)$$

Equations (55) and (56) give the required expressions of δ as a function of the characteristic elements of the strain and the stress¹².

A closed curve c' of the same type as c is now considered and the area of the band bounded by c and c' is denoted by $A_{cc'}$. Also denoted by a_{xx} , a_{yy} are the two astatic coordinates of the stress that acts on the band across the curves c and c' and is defined by

¹²If the state of strain is due to a simple plane dislocation on one of the boundaries, then $a_{xx} = a_{yy} = 0$ and it is sufficient to evaluate δ corresponding to a_{xx} or a_{yy} to obtain a simpler expression than equations (55) or (56).

$$\left. \begin{aligned} a_{xx} &= \int_c x X_n ds - \int_{c'} x X_n ds \\ a_{yy} &= \int_c y Y_n ds - \int_{c'} y Y_n ds \end{aligned} \right\} \quad (57)$$

When δ' denotes the expression of δ relative to c' , there is obtained from equations (56) and (57)

$$\begin{aligned} \delta - \delta' &= \frac{E}{8\pi k (1-\tau)} \left\{ \int_c \rho^2 \left(\frac{\partial D}{\partial x} n_x + \frac{\partial D}{\partial y} n_y \right) ds - 2 \int_c D (x n_x + y n_y) ds - \right. \\ &\quad \left. \int_{c'} \rho^2 \left(\frac{\partial D}{\partial x} n_x + \frac{\partial D}{\partial y} n_y \right) ds + 2 \int_{c'} D (x n_x + y n_y) ds \right\} + \frac{1}{2\pi} (a_{xx} + a_{yy}) \end{aligned} \quad (58)$$

When the transformation formula of line integrals into surface integrals is applied to equation (58) and the harmonic character of D is considered

$$\delta - \delta' = \frac{E}{2\pi k (\tau-1)} \int_{A_{cc'}} D dA_{cc'} + \frac{1}{2\pi} (a_{xx} + a_{yy}) \quad (59)$$

If \bar{I} denotes the mean value of the linear invariant of the stress I and the independence of δ on c in $A_{cc'}$ is accounted for, there is derived from equations (54) and (59)

$$\bar{I} = \frac{a_{xx} + a_{yy}}{A_{cc'}}$$

that is, the invariance of δ with respect to c expresses the property of the mean stress that assigns the mean value of the linear invariant of the stress¹³.

¹³For the properties of the mean of the stress mentioned above see reference 9.

Significance of a, b and of their invariance with respect to c. -
On the basis of equations (13) and (49),

$$\left. \begin{aligned} a &= \frac{1}{16\pi} \left\{ \int_c (x^2 - y^2) \frac{d\Delta F}{dn} ds - \int_c \Delta F \frac{d(x^2 - y^2)}{dn} ds \right\} \\ b &= \frac{1}{8\pi} \left\{ \int_c xy \frac{d\Delta F}{dn} ds - \int_c \Delta F \frac{dxy}{dn} ds \right\} \end{aligned} \right\} \quad (60)$$

When equations (53) and (54) are considered, equations (60) reduce to

$$\left. \begin{aligned} a &= \frac{E}{16\pi k} \left\{ \int_c (x^2 - y^2) d\omega - \frac{1}{1-\tau} \int_c \frac{d(x^2 - y^2)}{dn} D ds \right\} \\ b &= \frac{E}{8\pi k} \left\{ \int_c xy d\omega - \frac{1}{1-\tau} \int_c \frac{dxy}{dn} D ds \right\} \end{aligned} \right\} \quad (61)$$

or

$$\left. \begin{aligned} a &= \frac{E}{16\pi k(1-\tau)} \left\{ \int_c (x^2 - y^2) \frac{dD}{dn} ds - \int_c \frac{d(x^2 - y^2)}{dn} D ds \right\} \\ b &= \frac{E}{8\pi k(1-\tau)} \left\{ \int_c xy \frac{dD}{dn} ds - \int_c \frac{dxy}{dn} D ds \right\} \end{aligned} \right\} \quad (62)$$

With the aid of equations (61) and (62), a and b are thus expressed as functions of the rotation and of the coefficient of surface dilatation. It is also found (from equations (60)) that the invariance of a, b with respect to c is derived from a known property of harmonic functions.

On the basis of equation (1), the following expression may be written:

$$\int_c \psi(x, y) \frac{d\Delta F}{dn} ds = \int_c \psi(x, y) \left[\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) n_x + \left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} \right) n_y \right] ds \quad (63)$$

where $\psi(x,y)$ denotes any integrable function of x,y . From equation (63) there is obtained

$$\begin{aligned} \int_c \psi \frac{d\Delta F}{dn} ds - \int_{c'} \psi \frac{d\Delta F}{dn} ds &= \int_{A_{cc'}} \left[\frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial x} Y_y - \frac{\partial \psi}{\partial y} X_y \right) + \right. \\ &\left. \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial y} X_x - \frac{\partial \psi}{\partial x} X_y \right) \right] dA_{cc'} - \int_{A_{cc'}} \left[X_x \frac{\partial^2 \psi}{\partial y^2} + Y_y \frac{\partial^2 \psi}{\partial x^2} - 2X_y \frac{\partial^2 \psi}{\partial x \partial y} \right] dA_{cc'} + \\ &\int_{A_{cc'}} \psi \left[\frac{\partial^2 X_x}{\partial y^2} + \frac{\partial^2 Y_y}{\partial x^2} - 2 \frac{\partial^2 X_y}{\partial x \partial y} \right] dA_{cc'} \end{aligned} \quad (64)$$

Taking into account that equation of congruence of Saint-Venant which remains in effect in the case of plane systems expressed in terms of the components of the stress it is seen that the last integral of the second member of equation (64) is zero.

From equation (64) there is then obtained

$$\begin{aligned} \int_c \psi \frac{d\Delta F}{dn} ds - \int_{c'} \psi \frac{d\Delta F}{dn} ds &= \int_c \left[\left(\frac{\partial \psi}{\partial x} Y_y - \frac{\partial \psi}{\partial y} X_y \right) n_x + \right. \\ &\left. \left(\frac{\partial \psi}{\partial y} X_x - \frac{\partial \psi}{\partial x} X_y \right) n_y \right] ds - \int_{c'} \left[\left(\frac{\partial \psi}{\partial x} Y_y - \frac{\partial \psi}{\partial y} X_y \right) n_x + \right. \\ &\left. \left(\frac{\partial \psi}{\partial y} X_x - \frac{\partial \psi}{\partial x} X_y \right) n_y \right] ds - \int_{A_{cc'}} \left[X_x \frac{\partial^2 \psi}{\partial y^2} + Y_y \frac{\partial^2 \psi}{\partial x^2} - 2X_y \frac{\partial^2 \psi}{\partial x \partial y} \right] dA_{cc'} \end{aligned} \quad (65)$$

By taking into account equations (1) and (3), the following expression is obtained from equations (60) and (65) for $\psi = x^2 - y^2$.

$$a - a' = \frac{1}{8\pi} \left\{ \int_c (yY_n - xX_n) ds - \int_{c'} (yY_n - xX_n) ds - \int_{A_{cc'}} (Y_y - X_x) dA_{cc'} \right\} \quad (66)$$

where evidently a' denotes the expression of a in correspondence with c' .

By taking into account equation (57) and denoting by \bar{X}_x , \bar{Y}_y the mean values of X_x , Y_y in $A_{cc'}$, the following expression may be derived from the invariance of a and from equation (66):

$$\bar{X}_x - \bar{Y}_y = \frac{\alpha_{xx} - \alpha_{yy}}{A_{cc'}}$$

from which it is seen that the invariance of a with respect to c expresses one of the properties of the mean stress.

From equations (60) and (65), for $\psi = xy$ there is obtained, on the basis of equations (1) and (3)

$$b - b' = \frac{1}{8\pi} \left\{ - \int_c (xY_n + yX_n) ds + \int_{c'} (xY_n + yX_n) ds + 2 \int_{A_{cc'}} X_y dA_{cc'} \right\} \quad (67)$$

where the meaning of b' is clear.

The astatic coordinate of diverse indices of the stress acting on the band bounded by c, c' across its boundaries is denoted by α_{xy} and the mean value of X_y in $A_{cc'}$ is denoted by \bar{X}_y . From the invariance of b and from equation (67) there is immediately obtained, if the second fundamental equations of statics are taken into account,

$$\bar{X}_y - \frac{\alpha_{xy}}{A_{cc'}} = 0$$

It is therefore seen that the invariance of b with respect to c expresses that property of the mean stress which assigns the mean value of X_y .

Remark. If c is assumed to coincide with a circumference¹⁴ of center O and of radius R , equation (55) becomes

$$\delta = \frac{ER^2}{8\pi k} \left[\omega^* - \frac{2}{1-\tau} \int_0^{2\pi} D \, d\theta \right] + \frac{1}{2\pi} (a_{xx} + a_{yy}) \quad (68)$$

and presents δ as a linear combination of the increment of the rotation, because of a turn about the hole, of the mean value \bar{D} of D on the circumference c and of two astatic coordinates. In particular if $\omega^* = 0$ equation (68) becomes

$$\delta = - \frac{R^2}{2} \frac{D \bar{E}}{k(1-\tau)} - a_{xx} - a_{yy} \quad (69)$$

and: In the absence of dislocations and in the presence of dislocations that permit single-valued rotation ($\alpha = 0$), δ is equal to a linear combination of the mean value of the coefficient of surface dilatation along each circumference of center O and of two of the astatic coordinates of the stress that is transmitted across the circumference. This linear combination is therefore invariant with respect to the circumference.

In particular for a plane system with circular hole subject to a dislocation that leaves the rotation single-valued ($\alpha = 0$), it is sufficient from the fact that c is the circumference which bounds the hole and that correspondingly $a_{xx} = a_{yy} = 0$ to derive the result that δ expresses, except for an obvious coefficient, the mean value on c of the surface dilatation.

The same relation holds in the absence of dislocation, provided the external stress applied at the boundary of the hold satisfies the condition $a_{xx} = a_{yy} = 0$ (in particular if the hole is free from external stress).

¹⁴Naturally for the validity of the remark it is required that there exist a circumference entirely internal to T . If this is not the case, the biharmonic function F does not lose significance in a region T' comprising T in which circumferences can be drawn; the remark retains its validity provided the elastic system is considered extended in T' .

8. General Expressions of Components of Displacement in Doubly Connected Field.

From the preceding developments, it is easy to obtain the expressions for the components u, v of the displacement. There is thus the possibility of considering problems in which the displacements are assigned on the boundary or mixed problems.

For simplicity, the case of a doubly connected system will be considered, but it is not difficult to extend the results obtained to systems of any order of connectivity.

The values assumed by u, v are denoted by \bar{u}, \bar{v} when, in equations (26) and (27), φ_0, φ_1 are assumed to be zero in addition to δ , a , and b . The expressions of \bar{u}, \bar{v} are obtained from equation (22) by substituting in them $F - \varphi_0 - \varphi_1 - \delta \log \rho - a \cos 2\theta - b \sin 2\theta$ in place of F . When a rigid displacement is singled out and expression (26) is assumed for F itself, there is obtained in case (a):

$$\left. \begin{aligned} \bar{u} &= \frac{1}{2\pi} \left\{ l \left[\theta + \frac{1+\tau}{4} \sin 2\theta \right] + m \left[\frac{1-\tau}{2} \log \rho - \frac{1+\tau}{4} \cos 2\theta + \frac{1-\tau}{4} \right] + \right. \\ &\quad \left. r\rho \left[\theta \sin \theta + \frac{\tau-1}{2} \cos \theta \log \rho + \frac{\tau+1}{4} \cos \theta \right] \right\} \\ \bar{v} &= \frac{1}{2\pi} \left\{ l \left[\frac{\tau-1}{2} \log \rho - \frac{\tau+1}{4} \cos 2\theta + \frac{\tau-1}{4} \right] + m \left[\theta - \frac{\tau+1}{4} \sin 2\theta \right] + \right. \\ &\quad \left. r\rho \left[-\theta \cos \theta + \frac{\tau-1}{2} \sin \theta \log \rho + \frac{\tau+1}{4} \sin \theta \right] \right\} \end{aligned} \right\}$$

and in case (b) when the expression (27) is assumed for F ,

$$\left. \begin{aligned} \bar{u} &= \frac{k}{8\pi E} \left\{ X \left[2(\tau^2 - 2\tau - 3) \log \rho + (1+\tau)^2 \cos 2\theta + (1-\tau)^2 - 2(3+\tau) \right] + \right. \\ &\quad \left. Y(1+\tau)^2 \sin 2\theta - 4(1+\tau) M \frac{\sin \theta}{\rho} \right\} \\ \bar{v} &= \frac{k}{8\pi E} \left\{ X(1+\tau)^2 \sin 2\theta + Y \left[2(\tau^2 - 2\tau - 3) \log \rho - (1+\tau)^2 \cos 2\theta + \right. \right. \\ &\quad \left. \left. (1-\tau)^2 - 2(3+\tau) \right] + 4(1+\tau) M \frac{\cos \theta}{\rho} \right\} \end{aligned} \right\}$$

Next, if u', v' denote the values assumed by u, v when F is identified with $\delta \log \rho + a \cos 2\theta + b \sin 2\theta$, there is obtained on the basis of equation (22)

$$\begin{aligned} u' &= \frac{k}{E} \left\{ -\delta(1+\tau) \cos \theta + 4a \cos \theta (\cos^2 \theta - \tau \sin^2 \theta) + 2b [(3+\tau) \cos^2 \theta + (1-\tau) \sin^2 \theta] \sin \theta \right\} \frac{1}{\rho} \\ v' &= \frac{k}{E} \left\{ -\delta(1+\tau) \sin \theta + 4a \sin \theta (\tau \cos^2 \theta - \sin^2 \theta) + 2b [(3+\tau) \sin^2 \theta + (1-\tau) \cos^2 \theta] \cos \theta \right\} \frac{1}{\rho} \end{aligned}$$

Finally, if u'', v'' denote the values assumed by u, v when for F there is assumed $\varphi_0 + \varphi_1$,

$$\left. \begin{aligned} u &= \bar{u} + u' + u'' \\ v &= \bar{v} + v' + v'' \end{aligned} \right\}$$

where u'', v'' are given by equation (22) for $F = \varphi_0 + \varphi_1$.

It may be useful in many problems to assume for φ_0 and φ_1 a special series expansion in trigonometric binomials. By known theorems, the biharmonic functions φ_0, φ_1 can be developed into uniformly converging series, the first at each point at infinity, and the second for $\rho > 0$

$$\left. \begin{aligned} \varphi_0 &= \sum_{n=0}^{\infty} \rho^n [T_{1n} + \rho^2 T_{2n}] \\ \varphi_1 &= \sum_{n=3}^{\infty} \frac{1}{\rho^n} [T_{3n} + \rho^2 T_{4n}] + \frac{T_{31}}{\rho} + \frac{T_{32}}{\rho^2} \end{aligned} \right\} \quad (70)$$

where

$$T_{in} = a_{in} \cos n\theta + b_{in} \sin n\theta \quad (i = 1, 2, 3, 4; n = 0, 1, \dots) \quad (71)$$

In the series (70.2), the term T_{42} of the type $a \cos 2\theta + b \sin 2\theta$, already separately considered, has been suppressed. It is sufficient to substitute equation (70) into equation (22) to obtain

$$\left. \begin{aligned} u'' &= u''_{20} + \sum_{i=1}^3 u''_{i1} + \sum_{i=1, \dots, 4}^{n=2, 3, \dots *} u''_{in} \\ v'' &= v''_{20} + \sum_{i=1}^3 v''_{i1} + \sum_{i=1, \dots, 4}^{n=2, 3, \dots *} v''_{in} \end{aligned} \right\} \quad (72)$$

with

$$\left. \begin{aligned}
 &\begin{cases} u''_{1n} = -\frac{k}{E} (\tau+1) \rho^{n-1} \left[n T_{1n} \cos \theta - \sin \theta \frac{d T_{1n}}{d \theta} \right] \\ v''_{1n} = -\frac{k}{E} (\tau+1) \rho^{n-1} \left[n T_{1n} \sin \theta + \cos \theta \frac{d T_{1n}}{d \theta} \right] \end{cases} & (n = 1, 2, \dots) \\
 &\begin{cases} u''_{2n} = \frac{k}{E} \rho^{n+1} \left\{ \left[\frac{4}{n} + (\tau+1) \right] \sin \theta \frac{d T_{2n}}{d \theta} - [4 - (n+2)(\tau+1)] T_{2n} \cos \theta \right. \\ v''_{2n} = -\frac{k}{E} \rho^{n+1} \left\{ \left[\frac{4}{n} + (\tau+1) \right] \cos \theta \frac{d T_{2n}}{d \theta} - [4 - (n+2)(\tau+1)] T_{2n} \sin \theta \right. \end{cases} & (n = 0, 1, 2, \dots) \\
 &\begin{cases} u''_{3n} = \frac{k}{E} (\tau+1) \rho^{-(n+1)} \left[n T_{3n} \cos \theta + \sin \theta \frac{d T_{3n}}{d \theta} \right] \\ v''_{3n} = \frac{k}{E} (\tau+1) \rho^{-(n+1)} \left[n T_{3n} \sin \theta - \cos \theta \frac{d T_{3n}}{d \theta} \right] \end{cases} & (n = 1, 2, \dots) \\
 &\begin{cases} u''_{4n} = \frac{k}{E} \rho^{1-n} \left\{ \left[-\frac{4}{n} + (\tau+1) \right] \sin \theta \frac{d T_{4n}}{d \theta} + [4 - (2-n)(\tau+1)] \cos \theta T_{4n} \right. \\ v''_{4n} = \frac{k}{E} \rho^{1-n} \left\{ \left[\frac{4}{n} - (\tau+1) \right] \cos \theta \frac{d T_{4n}}{d \theta} + [4 - (2-n)(\tau+1)] \sin \theta T_{4n} \right. \end{cases} & (n = 3, 4, \dots)
 \end{aligned} \right\} (73)$$

The asterisk on the summation signs denote that the term of index 4,2 is not considered. It is necessary moreover to consider that for $n = 0$ the terms that contain $\frac{dT_{20}}{d\theta}$ are suppressed in equation (73).

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